Viscous relativistic hydrodynamics*

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Key references:

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Starting point: The conservation laws

$$\partial_\mu N^\mu = 0$$ charge conservation

$$\partial_\mu T^{\mu\nu} = 0$$ energy-momentum conservation

$$\partial_\mu S^\mu \geq 0$$ 2^{nd} law of thermodynamics

Ideal fluid decomposition

Ideal fluid dynamics $\Longleftrightarrow$ local thermal equilibrium $f(x,p) = f_{\rm eq}(x,p)$
$\Longleftrightarrow$ collision time scale $\ll$ macroscopic time scales

$$
\begin{aligned}
N^\mu &= \int \frac{d^3p}{E} p^\mu f(x,p) = n\, u^\mu && n = \text{(net) charge density} \\
T^{\mu\nu} &= \int \frac{d^3p}{E} p^\mu p^\nu f(x,p) && e = \text{energy density} \\
&= (e+p)\, u^\mu u^\nu - p\, g^{\mu\nu} && p = \text{pressure} \\
&= e\, u^\mu u^\nu - p\, \Delta^{\mu\nu} && \Delta^{\mu\nu} = g^{\mu\nu} - u^\mu u^\nu \\
S^\mu &= s\, u^\mu && s = \text{entropy density}
\end{aligned}
$$

First law of thermodynamics: $\quad Ts = p - \mu n + e$

$$\partial_\mu N^\mu = \partial_\mu T^{\mu\nu} = 0 \Longrightarrow \partial_\mu S^\mu = 0$$

(in absence of shock discontinuities, entropy is conserved)

Ideal fluid equations (in comoving frame)

Convective and transverse derivative:
$$\partial_\mu = u^\mu D + \nabla^\mu$$
$$D \equiv u^\nu \partial_\nu, \quad \nabla^\mu \equiv \Delta^{\mu\nu}\partial_\nu$$

$$\dot{n} = -n\,\theta$$
$$\dot{e} = -(e+p)\,\theta$$
$$\dot{u}^\mu = \frac{\nabla^\mu p}{e+p}$$

$\dot{f} = u^\mu \partial_\mu f \equiv Df =$ time derivative in local rest frame

$\theta \equiv \partial \cdot u =$ local expansion rate

$$p = p(n,e)$$

equation of state (EOS)

6 equations for 6 unknowns: $n,\ e,\ p,\ u^\mu$

Non-ideal fluid decomposition

$$f(x,p) = f_{\rm eq}(x,p) + \delta f(x,p)$$

$$\boxed{\begin{aligned} N^\mu &= n\, u^\mu + V^\mu \\ &= N^\mu_{\rm eq} + \delta N^\mu \\ T^{\mu\nu} &= e\, u^\mu u^\nu - p\Delta^{\mu\nu} \\ &\quad - \Pi\Delta^{\mu\nu} + \pi^{\mu\nu} \\ &\quad + W^\mu u^\nu + W^\nu u^\mu \\ &= T^{\mu\nu}_{\rm eq} + \delta T^{\mu\nu} \\ S^\mu &= s\, u^\mu + \Phi^\mu \\ &= S^\mu_{\rm eq} + \delta S^\mu \end{aligned}}$$

$n = u_\mu N^\mu$

$V^\mu = \Delta^{\mu\nu} N_\nu$ = charge flow in l.r.f.

$e = u_\mu T^{\mu\nu} u_\nu$

$\Pi = -\frac{1}{3}\Delta_{\mu\nu} T^{\mu\nu} - p$ = viscous bulk pressure

$W^\mu = u^\nu T_{\nu\lambda}\Delta^{\lambda\mu}$ = energy flow in l.r.f.

$= q^\mu + \frac{e+p}{n} V^\mu$ $\quad q^\mu$ = heat flow in l.r.f.

$\pi^{\mu\nu} = T^{\langle\mu\nu\rangle}$

$\equiv \left[\frac{1}{2}(\Delta^{\mu\sigma}\Delta^{\nu\tau} + \Delta^{\mu\tau}\Delta^{\nu\sigma}) - \frac{1}{3}\Delta^{\mu\nu}\Delta^{\tau\sigma}\right] T_{\tau\sigma}$

= viscous shear pressure tensor $(\pi^\mu_\mu = 0)$

$s = u_\mu S^\mu$

$\Phi^\mu = \Delta^{\mu\nu} S_\nu$ = entropy flow in l.r.f.

Frame choice and matching conditions

The local equilibrium distribution $f_{\rm eq}(x,p)$ (with local temperature $T(x)$ and chemical potential $\mu(x)$) that best matches the non-equilibrium $f(x,p)$ is defined by the matching conditions

$$\boxed{u_\mu\,\delta T^{\mu\nu}\,u_\nu = u_\mu\,\delta N^\mu = 0}$$

Local rest frame is ambiguous:

Eckart frame: $V^\mu = 0,\quad q^\mu = W^\mu$
Landau frame: $W^\mu = 0,\ q^\mu = -\frac{e+p}{n} V^\mu$

(Intermediate frames also possible.)

$\Longrightarrow$ Need $1+3+5=9$ additional equations for Π, q^μ, $\pi^{\mu\nu}$ from underlying transport theory.

Non-ideal fluid equations

$$
\begin{aligned}
\dot{n} &= -n\,\theta - \nabla\cdot V + V\cdot\dot{u} \\
\dot{e} &= -(e+p+\Pi)\,\theta + \pi_{\mu\nu}\nabla^{\langle\mu}u^{\nu\rangle} - \nabla\cdot W + 2\,W\cdot\dot{u} \\
(e{+}p{+}\Pi)\,\dot{u}^\mu &= \nabla^\mu(p+\Pi) - \Delta^{\mu\nu}\nabla^\sigma\pi_{\nu\sigma} + \pi^{\mu\nu}\dot{u}_\nu \\
&\quad -[\Delta^{\mu\nu}\dot{W}_\nu + W^\mu\theta + (W\cdot\nabla)u^\mu]
\end{aligned}
$$

Depending on frame, can set either $V^\mu=0$ or $W^\mu=0$. In Landau frame ($W^\mu=0$) and for baryon-free systems ($n=0$, no heat conduction) equations simplify to:

$$
\begin{aligned}
\dot{e} &= -(e+p+\Pi)\,\theta + \pi_{\mu\nu}\nabla^{\langle\mu}u^{\nu\rangle} \\
(e{+}p{+}\Pi)\,\dot{u}^\mu &= \nabla^\mu(p+\Pi) - \Delta^{\mu\nu}\nabla^\sigma\pi_{\nu\sigma} + \pi^{\mu\nu}\dot{u}_\nu
\end{aligned}
$$

Need extra equations for bulk and shear viscous pressures Π, $\pi^{\mu\nu}$.

Follow Chapman-Enskog strategy: write $f(x,p) = f_{\rm eq}\big(p\cdot u(x); T(x), \mu(x)\big) + \delta f(x,p)$ and assume that $\delta f \ll f$ (and thus δN^μ and $\delta T^{\mu\nu}$) can be expanded in gradients of equilibrium parameters T, μ, u_μ.

The second law of thermodynamics (I)

In equilibrium the identity $Ts = p - \mu n + e$ can be rewritten as

$$S^\mu_{\rm eq} = p(\alpha,\beta)\beta^\mu - \alpha N^\mu_{\rm eq} + \beta_\nu T^{\mu\nu}_{\rm eq}$$

where $\alpha \equiv \mu/T$, $\beta \equiv 1/T$, and $\beta^\mu \equiv u^\mu/T$.

The most general off-equilibrium generalization is (Israel & Stewart 1979)

$$S^\mu = p(\alpha,\beta)\beta^\mu - \alpha N^\mu + \beta_\nu T^{\mu\nu} + Q^\mu(\delta N^\mu, \delta T^{\mu\nu})$$

where Q^μ is second and higher order in the off-equilibrium deviations δN^μ and $\delta T^{\mu\nu}$.

The Gibbs-Duhem relation $dp = s\,dT + n\,d\mu$ can be recast as

$$\partial_\mu(p(\alpha,\beta)\beta^\mu) = N^\mu_{\rm eq}\partial_\mu\alpha - T^{\mu\nu}_{\rm eq}\partial_\mu\beta_\nu$$

Using also the conservation laws, the entropy production rate takes the form

$$\partial_\mu S^\mu = -\delta N^\mu\partial_\mu\alpha + \delta T^{\mu\nu}\partial_\mu\beta_\nu + \partial_\mu Q^\mu$$

The second law of thermodynamics (II)

In the Chapman-Enskog spirit, one now postulates linear relations between the off-equilibrium flows δN^μ, $\delta T^{\mu\nu}$ and the thermodynamic forces $\partial^\mu \alpha$, $\partial^{(\mu}\beta^{\nu)}$, consistent with the second law

$$\partial_\mu S^\mu = -\delta N^\mu \partial_\mu \alpha + \delta T^{\mu\nu}\partial_\mu \beta_\nu + \partial_\mu Q^\mu \geq 0$$

These relations depend on the choice of Q^μ. Standard dissipative relativistic fluid dynamics assumes $Q^\mu = 0$. In this case

$$T\partial_\mu S^\mu = \Pi X - q^\mu X_\mu + \pi^{\mu\nu} X_{\langle\mu\nu\rangle} \equiv \frac{\Pi^2}{\zeta} - \frac{q^\mu q_\mu}{2\lambda T} + \frac{\pi^{\alpha\beta}\pi_{\alpha\beta}}{2\eta} \geq 0,$$

with thermodynamic forces $X \equiv -\nabla \cdot u = -\theta$, $X^\mu \equiv \frac{\nabla^\mu T}{T} - \dot{u}^\mu = -\frac{nT}{e+p}\nabla^\mu \left(\frac{\mu}{T}\right)$ and $X_{\langle\mu\nu\rangle} \equiv \nabla_{\langle\mu} u_{\nu\rangle}$, can be satisfied by setting

$$\Pi = -\zeta\theta, \quad q^\mu = -\lambda \frac{nT^2}{e+p}\nabla^\mu \left(\frac{\mu}{T}\right), \quad \pi^{\mu\nu} = 2\eta\, \nabla^{\langle\mu} u^{\nu\rangle}$$

with positive transport coefficents $\zeta \geq 0$, $\lambda \geq 0$, and $\eta \geq 0$.
Unfortunately, plugging these equations for Π, q^μ, and $\pi^{\mu\nu}$ directly into the non-ideal hydro equations leads to acausal signal propagation.

The second law of thermodynamics (III)

Causal relativistic fluid dynamics requires keeping Q^μ in the entropy flux, at least up to terms of second order in the irreversible flows.

$$S^\mu = s u^\mu + \frac{q^\mu}{T} + Q^\mu$$

Setting $q^\nu = 0$ ($n = 0$) for simplicity, we get up to second order

$$S^\mu = s u^\mu - (\beta_0 \Pi^2 + \beta_2 \pi_{\nu\lambda} \pi^{\nu\lambda}) \frac{u^\mu}{2T}$$

This yields (after some algebra)

$$\begin{aligned}
T \partial_\mu S^\mu \quad &= \quad \Pi \left[-\theta - \beta_0 \dot{\Pi} - \Pi \, T \partial_\mu \left(\frac{\beta_0 u^\mu}{2T} \right) \right] \\
&+ \quad \pi^{\alpha\beta} \left[\nabla_{\langle\alpha} u_{\beta\rangle} - \beta_2 \dot{\pi}_{\alpha\beta} - \pi_{\alpha\beta} \, T \partial_\mu \left(\frac{\beta_2 u^\mu}{2T} \right) \right] \\
&\overset{!}{=} \quad \frac{\Pi^2}{\zeta} + \frac{\pi^{\alpha\beta} \pi_{\alpha\beta}}{2\eta} \geq 0
\end{aligned}$$

The thermodynamic forces $-\theta$, $\nabla_{\langle\alpha} u_{\beta\rangle}$ are seen to be self-consistently modified by the irreversible flows Π, $\pi_{\alpha\beta}$. This leads to dynamical ("transport") equations for Π, $\pi_{\alpha\beta}$.

Transport equations for the irreversible flows

The resulting transport equations for Π, $\pi_{\alpha\beta}$ are

$$
\begin{aligned}
\dot{\Pi} &= -\frac{1}{\tau_\Pi}\left[\Pi + \zeta\theta + \Pi\zeta T \partial_\mu \left(\frac{\tau_\Pi u^\mu}{2\zeta T}\right)\right] \approx -\frac{1}{\tau_\Pi}\left[\Pi + \zeta\theta\right] \\
\Delta_{\alpha\mu}\Delta_{\beta\nu}\dot{\pi}^{\mu\nu} &= -\frac{1}{\tau_\pi}\left[\pi_{\alpha\beta} - 2\zeta\nabla_{\langle\alpha}u_{\beta\rangle} + \pi_{\alpha\beta}\eta T \partial_\mu \left(\frac{\tau_\pi u^\mu}{2\eta T}\right)\right] \\
\Longrightarrow \quad \dot{\pi}_{\alpha\beta} &\approx -\frac{1}{\tau_\pi}\left[\pi_{\alpha\beta} - 2\zeta\nabla_{\langle\alpha}u_{\beta\rangle}\right] - (u_\alpha\pi_{\beta\nu} + u_\beta\pi_{\alpha\nu})\dot{u}^\nu
\end{aligned}
$$

where we introduced the relaxation times $\tau_\Pi = \zeta\beta_0$, $\tau_\pi = 2\eta\beta_2$. In principle, both ζ, η and τ_Π, τ_π should be calculated from the underlying kinetic theory.

The purple terms are of second order in the derivatives of the thermodynamic equilibrium quantities and, in the regime of validity of the approach, should be neglected relative to the other terms.

[Keeping them would require also keeping third-order terms in the entropy flow Q^μ for consistency, and would modify both the effective local relaxation times $\tau_{\Pi,\pi}$ and the viscosities η, ζ by amounts which depend on the local hydrodynamic expansion rate.]

Viscous relativistic hydrodynamics (Israel & Stewart 1979)

Include shear viscosity η, neglect bulk viscosity (massless partons) and heat conduction ($\mu_B \approx 0$); solve

$$\boxed{\partial_\mu T^{\mu\nu} = 0}$$

with modified energy momentum tensor

$$\boxed{T^{\mu\nu}(x) = \big(e(x)+p(x)\big)u^\mu(x)u^\nu(x) - g^{\mu\nu}p(x) + \pi^{\mu\nu}.}$$

$\pi^{\mu\nu}$ = traceless viscous pressure tensor which relaxes locally to 2η times the shear tensor $\nabla^{\langle\mu}u^{\nu\rangle}$ on a microscopic kinetic time scale τ_π:

$$\boxed{D\pi^{\mu\nu} = -\frac{1}{\tau_\pi}\left(\pi^{\mu\nu} - 2\eta\nabla^{\langle\mu}u^{\nu\rangle}\right) - \left(u^\mu\pi^{\nu\lambda}+u^\nu\pi^{\mu\lambda}\right)Du_\lambda}$$

where $D \equiv u^\mu\partial_\mu$ is the time derivative in the local rest frame.

Kinetic theory relates η and τ_π, but for a strongly coupled QGP neither η nor this relation are known $\Longrightarrow$ treat η and τ_π as independent phenomenological parameters. For consistency: $\boxed{\tau_\pi\theta \ll 1}$ ($\theta = \partial^\mu u_\mu =$ local expansion rate).

(1+1)-d viscous hydrodynamic equations

(Muronga & Rischke 2004, Chaudhuri & Heinz 2005)

[For (2+1)-d viscous hydrodynamic equations see Heinz, Song & Chaudhuri, nucl-th/0510014]

Azimuthally symmetric transverse dynamics with long. boost invariance:
Use (τ, r, ϕ, η) coordinates and solve

- hydrodynamic equations for $T^{\tau\tau} = (e+\mathcal{P})\gamma_r^2 - \mathcal{P}, \quad T^{\tau r} = (e+\mathcal{P})\gamma_r^2 v_r$
  (with "effective pressure" $\mathcal{P} = p - r^2\pi^{\phi\phi} - \tau^2\pi^{\eta\eta}$) together with
- kinetic relaxation equations for $\pi^{\phi\phi},\ \pi^{\eta\eta}$:

$$\frac{1}{\tau}\partial_\tau\Big(\tau T^{\tau\tau}\Big) + \frac{1}{r}\partial_r\Big(r(T^{\tau\tau} + \mathcal{P})v_r\Big) = -\frac{p + \tau^2\pi^{\eta\eta}}{\tau},$$

$$\frac{1}{\tau}\partial_\tau\Big(\tau T^{\tau r}\Big) + \frac{1}{r}\partial_r\Big(r(T^{\tau r}v_r + \mathcal{P})\Big) = +\frac{p + r^2\pi^{\phi\phi}}{r},$$

$$\Big(\partial_\tau + v_r\partial_r\Big)\pi^{\eta\eta} = -\frac{1}{\gamma_r\tau_\pi}\left[\pi^{\eta\eta} - \frac{2\eta}{\tau^2}\left(\frac{\theta}{3} - \frac{\gamma_r}{\tau}\right)\right],$$

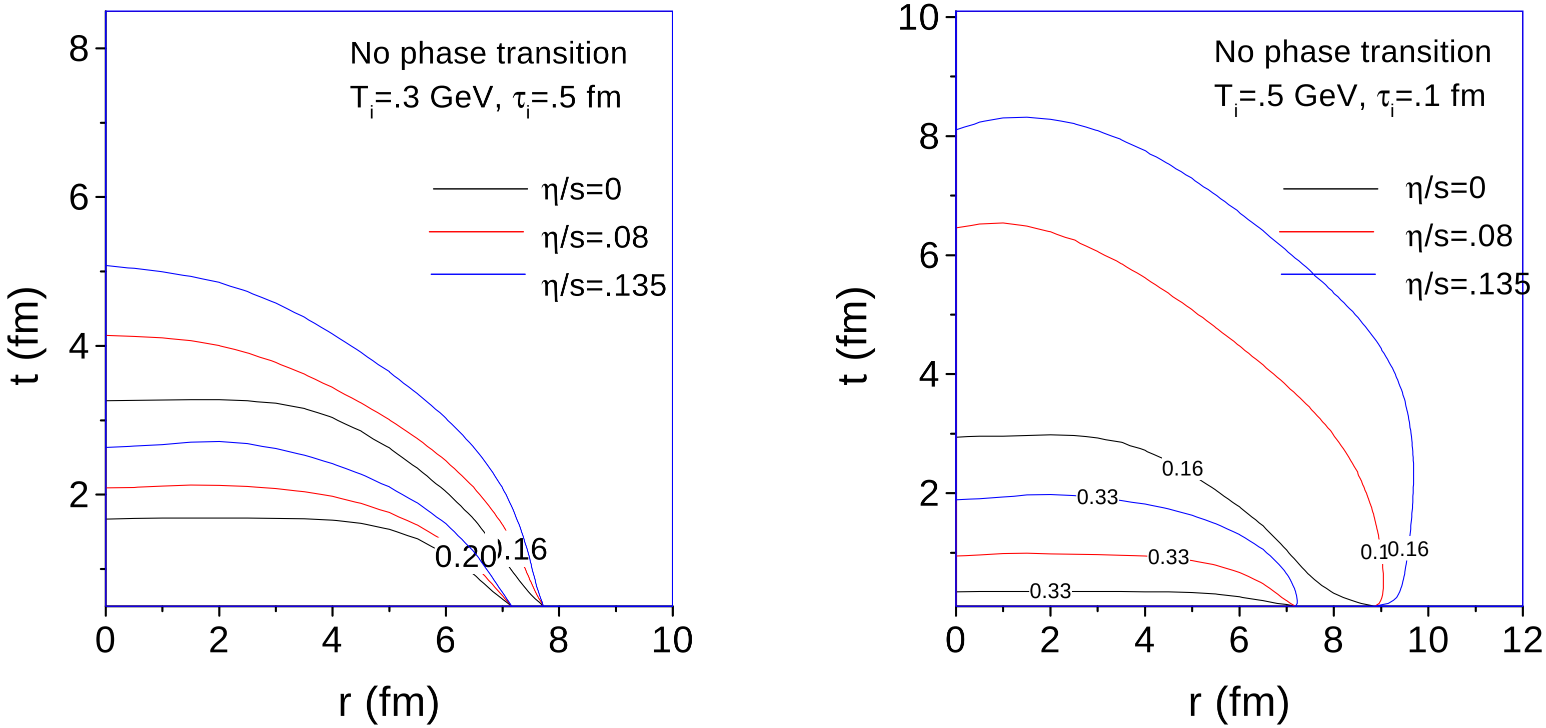
$$\Big(\partial_\tau + v_r\partial_r\Big)\pi^{\phi\phi} = -\frac{1}{\gamma_r\tau_\pi}\left[\pi^{\phi\phi} - \frac{2\eta}{r^2}\left(\frac{\theta}{3} - \frac{\gamma_r v_r}{r}\right)\right].$$

Close equations with EOS $p(e)$ where $e = T^{\tau\tau} - v_r T^{\tau r}$ and $v_r = T^{\tau r}/(T^{\tau\tau} + \mathcal{P})$.

(1+1)-d viscous hydrodynamics: first results (I)

(Chaudhuri & Heinz, nucl-th/0504022)

Viscosity effects on freeze-out surface ($\tau_\pi = \frac{3\eta}{2p}$, $\pi^{rr}_{\text{ini}} = \frac{2\eta}{3\tau_i}$):



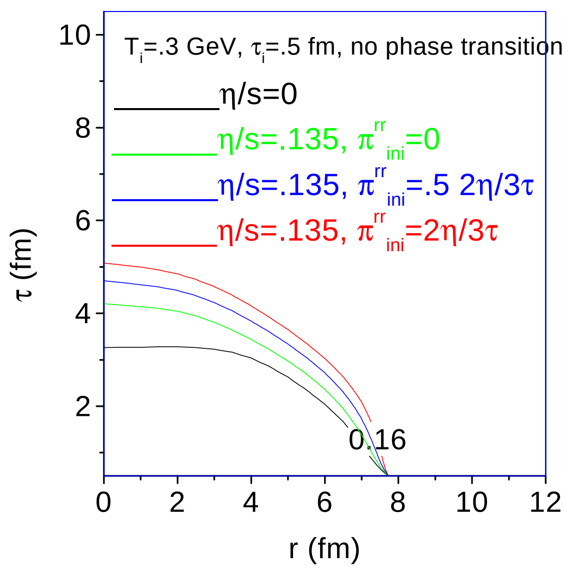
- Both sets of initial conditions have similar initial total entropy.
- Viscosity slows down cooling and gives more time for transverse expansion.
- Viscous effects are larger for smaller τ_i, due to faster initial expansion rate.

(1+1)-d viscous hydrodynamics: first results (II)

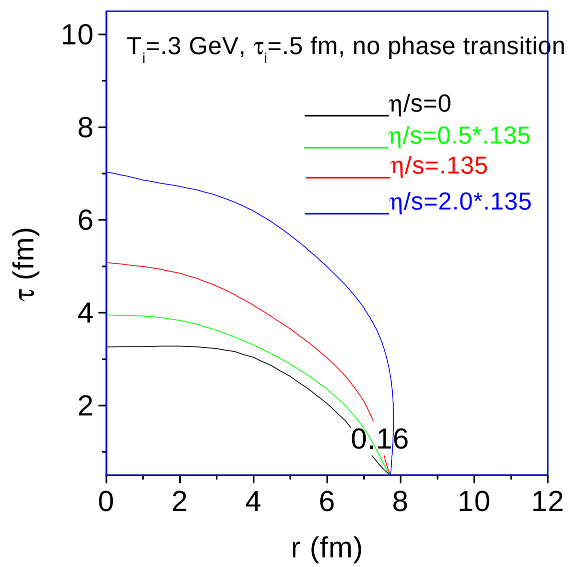
(Chaudhuri & Heinz, nucl-th/0504022)

Sensitivity to initial π^{rr}, $\frac{\eta}{s}$, and relaxation time τ_π ($T_{\rm f} = 160\,$MeV):

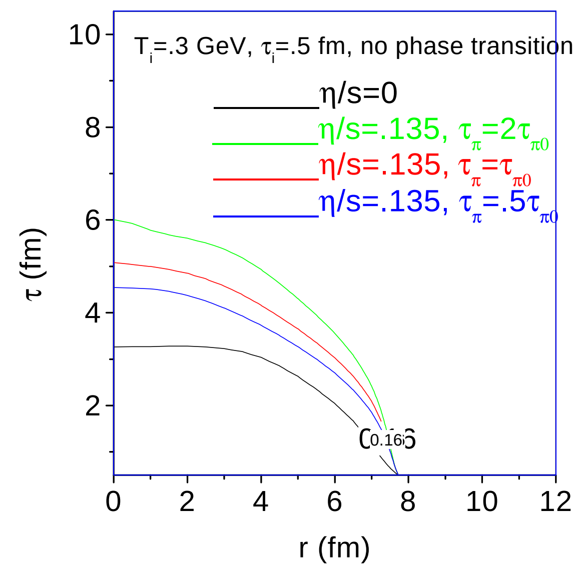
$\tau_\pi = \frac{3\eta}{2p},\ \frac{\eta}{s} = 0.135$



$\tau_\pi = \frac{3\eta}{2p},\ \pi^{rr}_{\rm ini} = \frac{2\eta}{3\tau_i}$



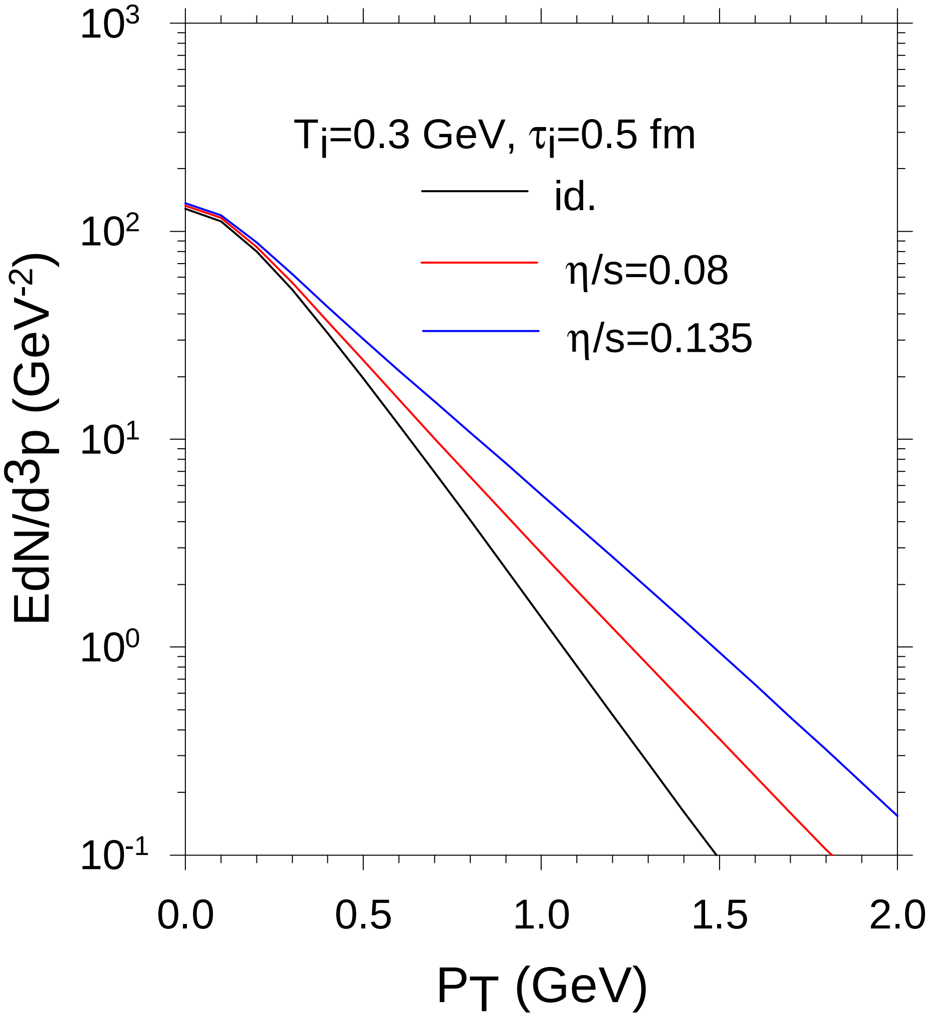
$\eta/s = 0.135,\ \pi^{rr}_{\rm ini} = \frac{2\eta}{3\tau_i}$



- Larger initial viscous pressures create larger overall viscous effects ("memory effect")
- Significant viscous effects for $\frac{\eta}{s} > \frac{\hbar}{4\pi}$
- At fixed $\frac{\eta}{s}$, viscous effects increase with increasing relaxation time τ_π

(1+1)-d viscous hydrodynamics: first results (III)

(Chaudhuri & Heinz, nucl-th/0504022)



Viscous shear pressure reduces longitudinal work, but increases transverse flow

$\Longrightarrow$ same initial conditions yield flatter transverse momentum spectra than for ideal fluid dynamics

Summary

- Causal relativistic dissipative hydrodynamics requires solution of a coupled set of (i) hydrodynamic equations with additional irreversible flow corrections and (ii) kinetic relaxation equations for these irreversible flows.

- Relaxation equations for dissipative flows are derived from a second-order approach to implementing the 2^{nd} law of thermodynamics, keeping terms up to second order in derivatives of equilibrium quantities.

- For each non-vanishing space-time component of the hydro equations, we must solve one transport equation for each non-vanishing transport coefficient (bulk viscosity, shear viscosity, heat conduction) $\Longrightarrow$ significantly increased numerical complexity.

- (1+1)-dimensional viscous hydro is under investigation;
(2+1)-dimensional code is under construction.